Indirect Proofs

Outline for Today

What is an Implication?

- Understanding a key type of mathematical statement.

Negations and their Applications

- How do you show something is *not* true?

Proof by Contrapositive

- What's a contrapositive?
- And some applications!

Proof by Contradiction

- The basic method.
- And some applications!

Logical Implication

Implications

An ***implication*** is a statement of the form

If *P* is true, then *Q* is true.

Some examples:

- If n is an even integer, then n^2 is an even integer.
- If $A \subseteq B$ and $B \subseteq A$, then $A = B$.
- If you like the way you look that much, (ohhh baby) then you should go and love yourself.

Implications

An ***implication*** is a statement of the form

If *P* is true, then *Q* is true.

In the above implication, the statement "*P* is true" is called the ***antecedent*** and the statement "*Q* is true" is called the ***consequent***.

What Implications Mean

Consider the simple statement

If I put fire near cotton, it will burn.

Some questions to consider:

- Does this apply to all fire and all cotton, or just some types of fire and some types of cotton? *(Scope)*
- Does the fire cause the cotton to burn, or does the cotton burn for another reason? *(Causality)*
- These are significantly deeper questions than they might seem.
- To mathematically study implications, we need to formalize what implications really mean.

Understanding Implications

“If there's a rainbow in the sky, then it's raining somewhere.”

- In mathematics, implication is *directional*.
- The above statement doesn't mean that if it's raining somewhere, there has to be a rainbow.
- In mathematics, implications only say something about the consequent when the antecedent is true.
- If there's no rainbow, it doesn't mean there's no rain.
- In mathematics, implication says nothing about *causality*.
- Rainbows do not cause rain. ☺

What Implications Mean

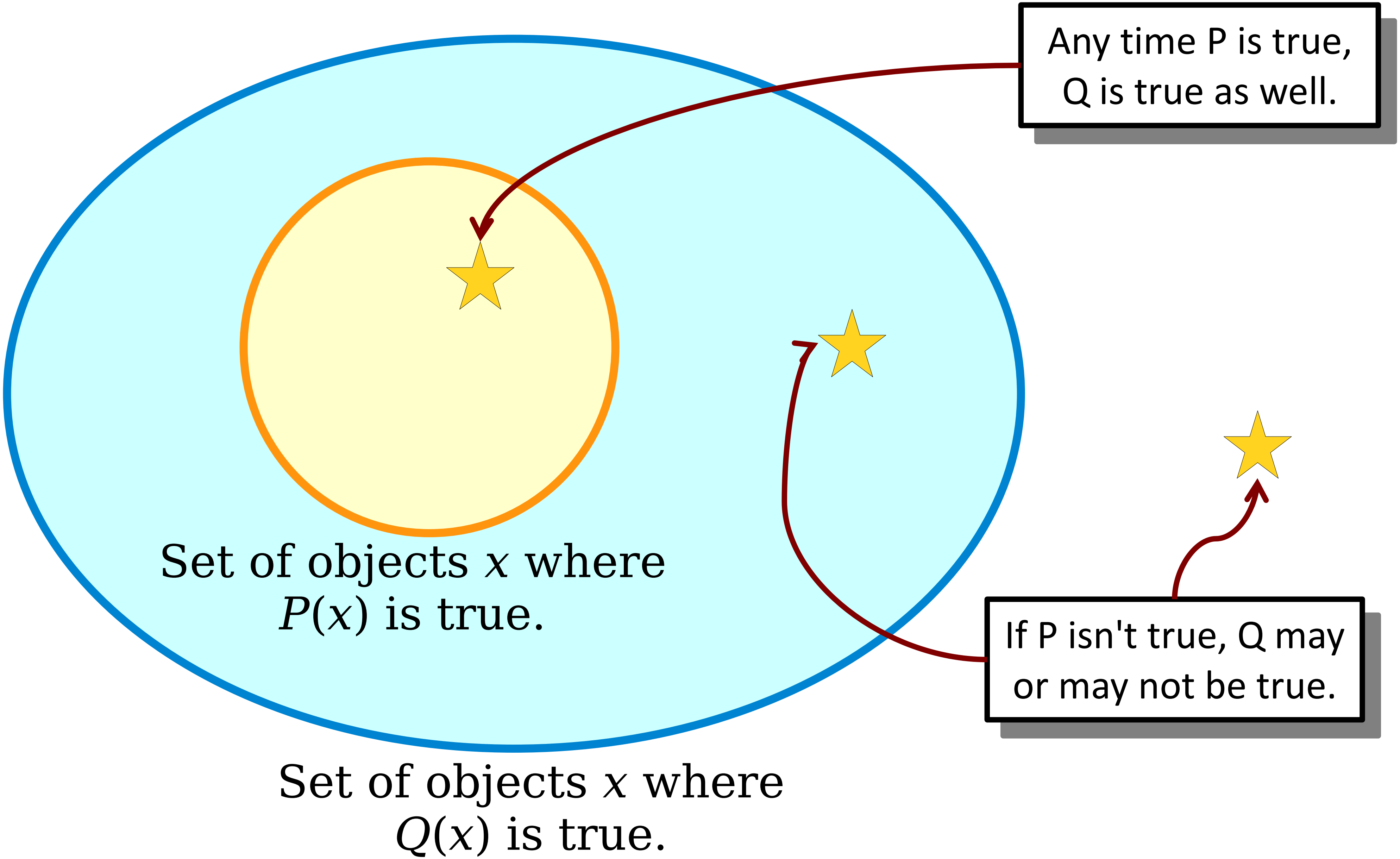
In mathematics, a statement of the form

For any x, if $P(x)$ is true, then $Q(x)$ is true

means that any time you find an object x where $P(x)$ is true, you will see that $Q(x)$ is also true (for that same x).

There is no discussion of causation here. It simply means that if you find that $P(x)$ is true, you'll find that $Q(x)$ is also true.

Implication, Diagrammatically



Negations

Negations

A ***proposition*** is a statement that is either true or false.

Some examples:

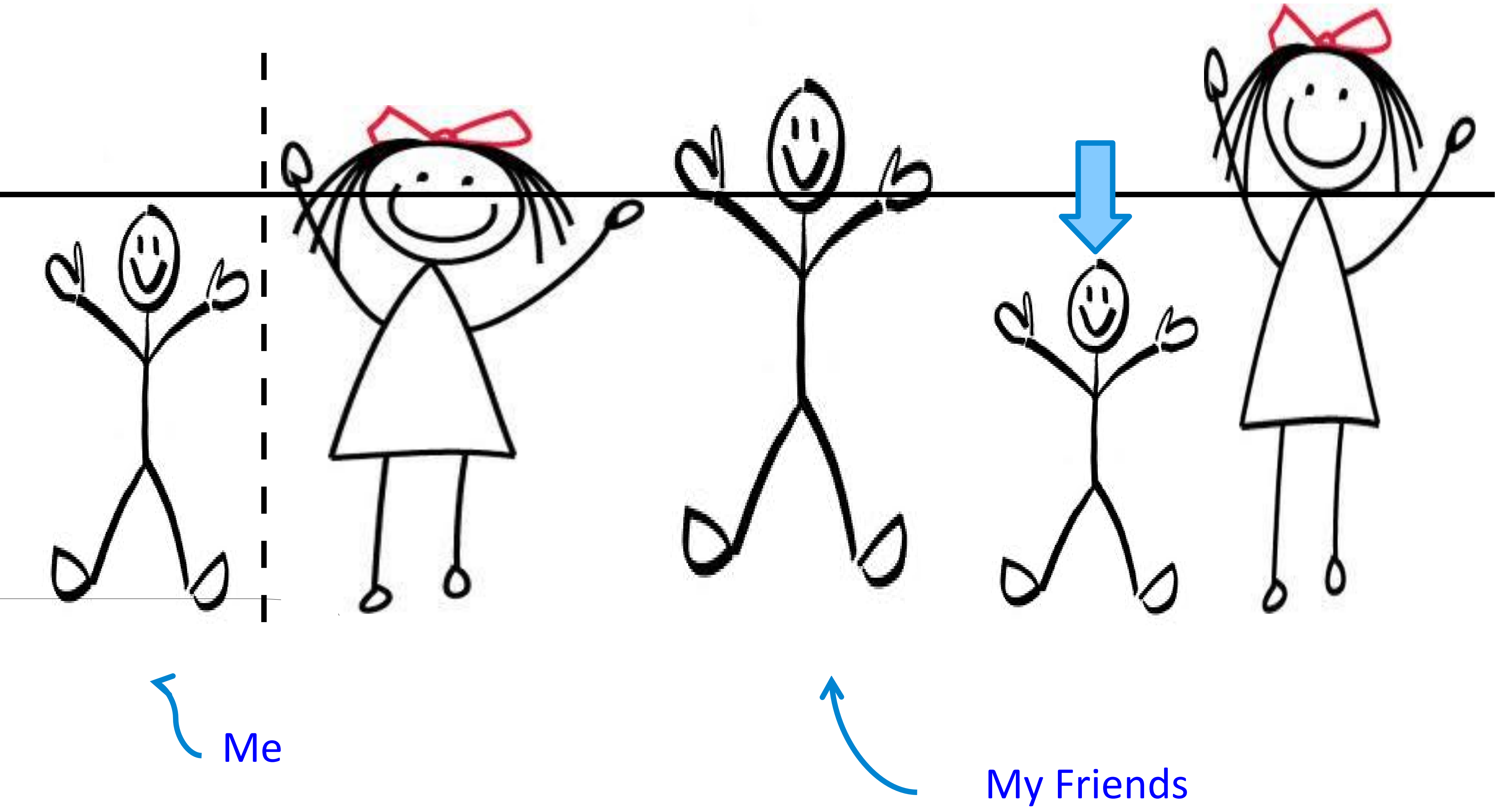
- If n is an even integer, then n^2 is an even integer.
- $\emptyset = \mathbb{R}$.
- The new me is still the real me.

The ***negation*** of a proposition X is a proposition that is true whenever X is false and is false whenever X is true.

- For example, consider the statement "it is snowing outside."
- Its negation is "it is not snowing outside."
- Its negation is *not* "it is sunny outside." ⚠
- Its negation is *not* "we're in the Bay Area." ⚠

How do you find the negation of a statement?

“All My Friends Are Taller Than Me”
How to negate this?



The negation of the ***universal*** statement

Every *P* is a *Q*

is the ***existential*** statement

There is a *P* that is not a *Q*.

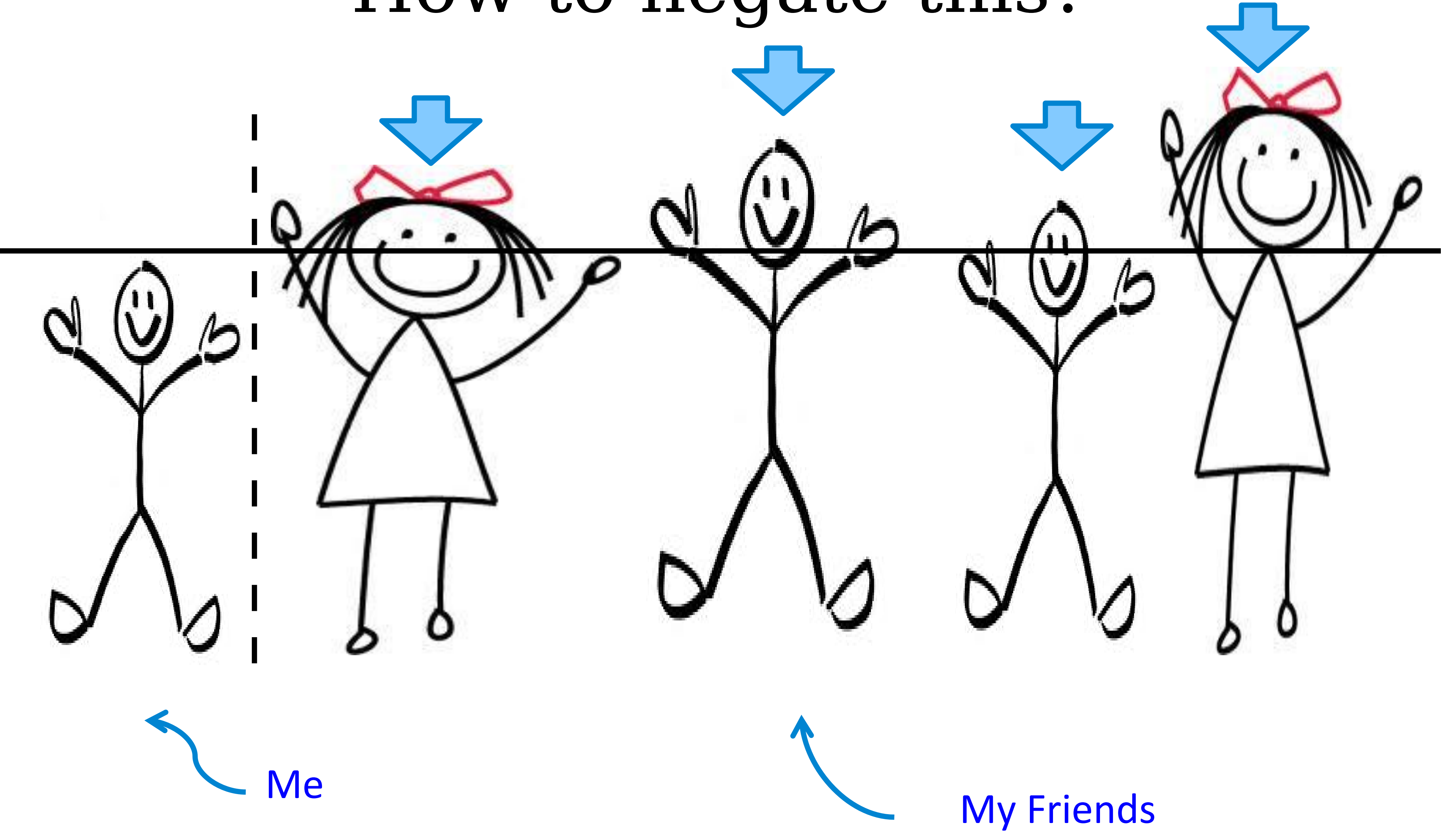
The negation of the ***universal*** statement

For all *x*, *P*(*x*) is true.

is the ***existential*** statement

There exists an *x* where *P*(*x*) is false.

“Some Friend Is Shorter Than Me”
How to negate this?



The negation of the ***existential*** statement

There exists a *P* that is a *Q*

is the ***universal*** statement

Every *P* is not a *Q*.

The negation of the ***existential*** statement

There exists an *x* where *P*(*x*) is true

is the ***universal*** statement

For all *x*, *P*(*x*) is false.

Puppy Logic

Consider the statement

If *x* is a puppy, then I love *x*.

The statement below is ***not*** its negation:

⚠ **If *x* is a puppy, then I don’t love *x*** ⚠

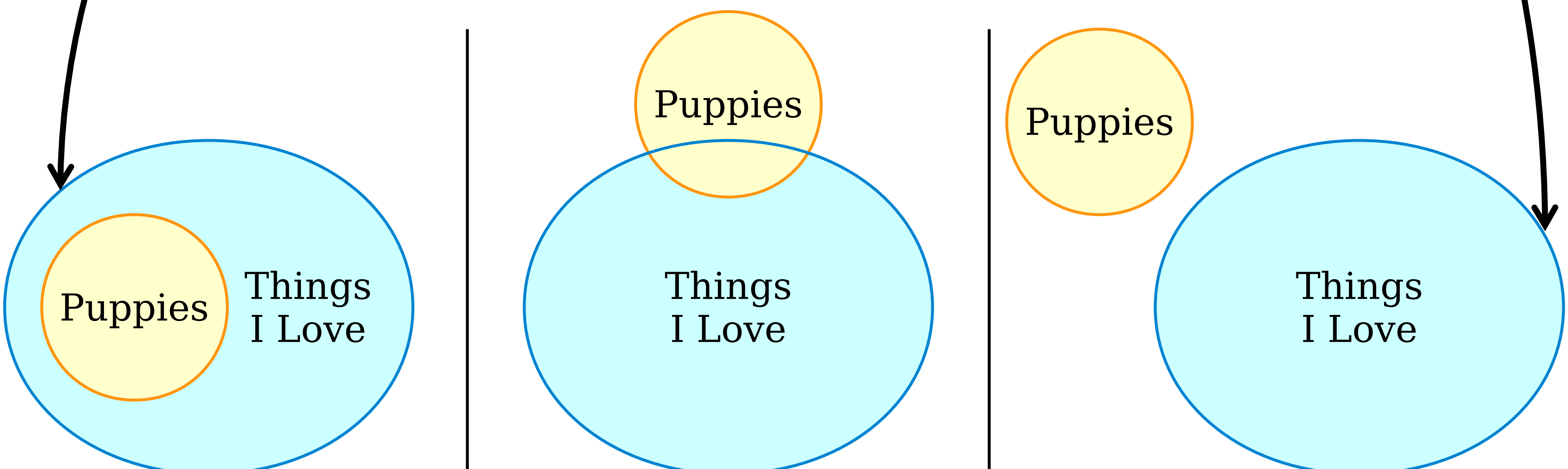
It might not be obvious, at a first glance, why these statements aren’t negations of one another. Talk this over with someone next to you and see if you can sort out why.

Puppy Logic

If *x* is a puppy, then I love *x*.

⚠ **If *x* is a puppy, then I don't love *x*** ⚠

So what is the proper negation?

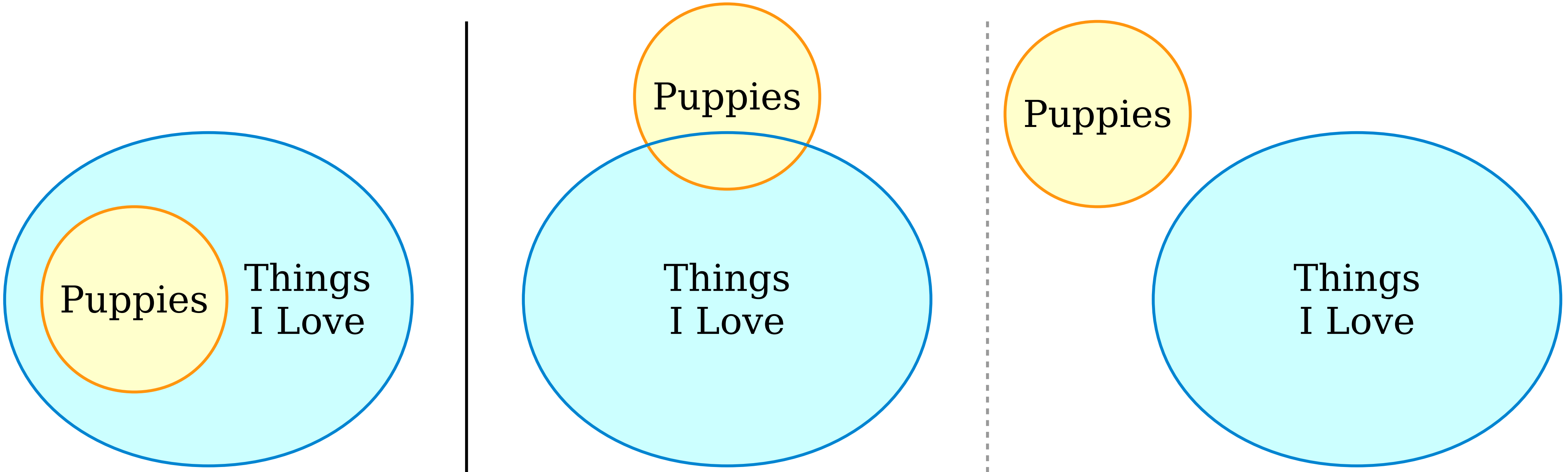


"I love *all* puppies." | **"It's complicated."** | **"I don't love *any* puppies."**

Puppy Logic

If *x* is a puppy, then I love *x*.

There is a puppy I don't love.



"I love *all* puppies."

"There is a puppy that I don't love."

The negation of the statement

“For any x, if $P(x)$ is true, then $Q(x)$ is true”

is the statement

“There is at least one x where $P(x)$ is true and $Q(x)$ is false.”

The negation of an implication is not an implication!

How to Negate Universal Statements:

"For all x, P(x) is true"

becomes

"There is an x where P(x) is false."

How to Negate Existential Statements:

"There exists an x where P(x) is true"

becomes

"For all x, P(x) is false."

How to Negate Implications:

"For every x, if $P(x)$ is true, then $Q(x)$ is true"

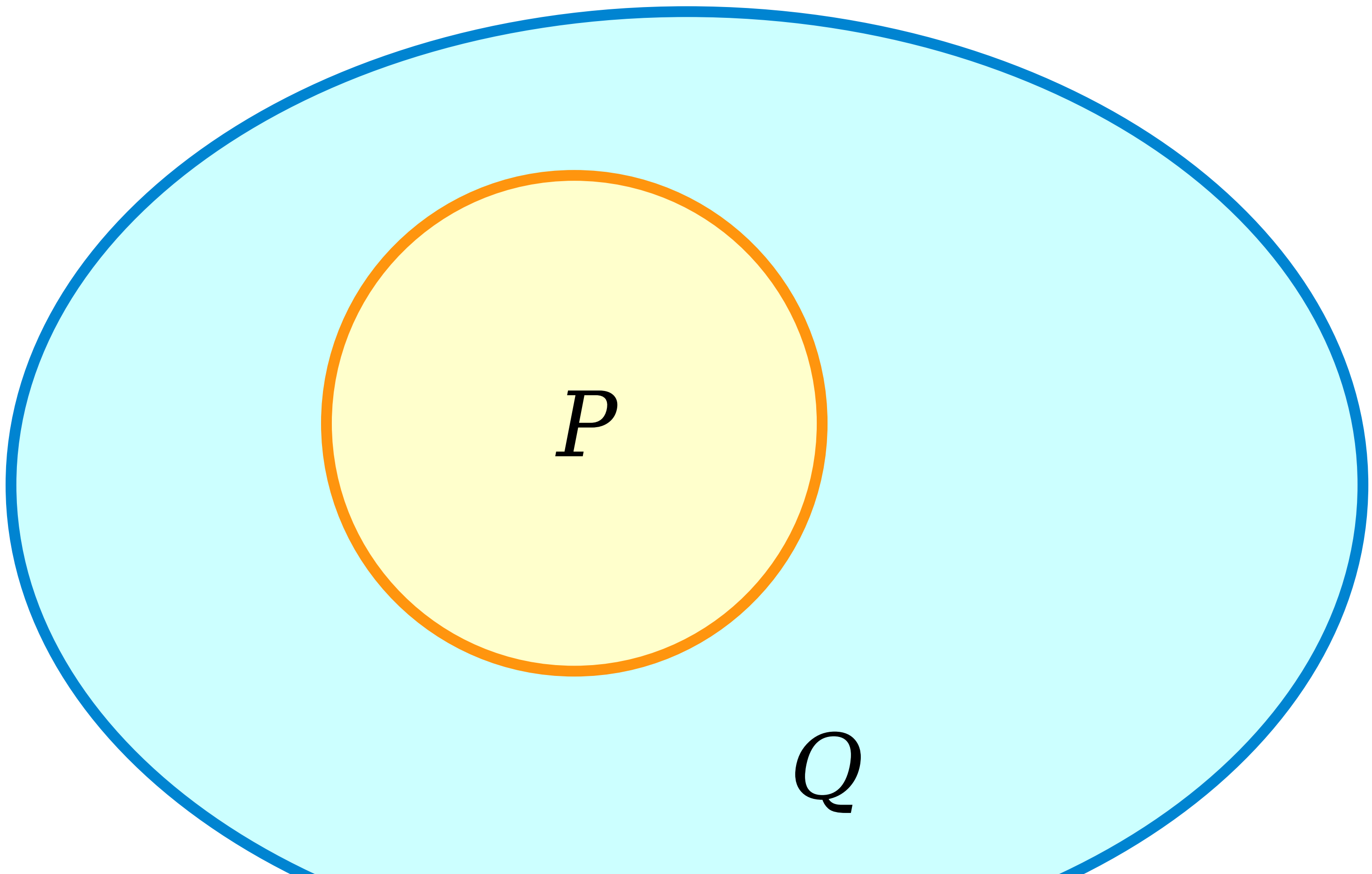
becomes

"There is an x where $P(x)$ is true and $Q(x)$ is false"

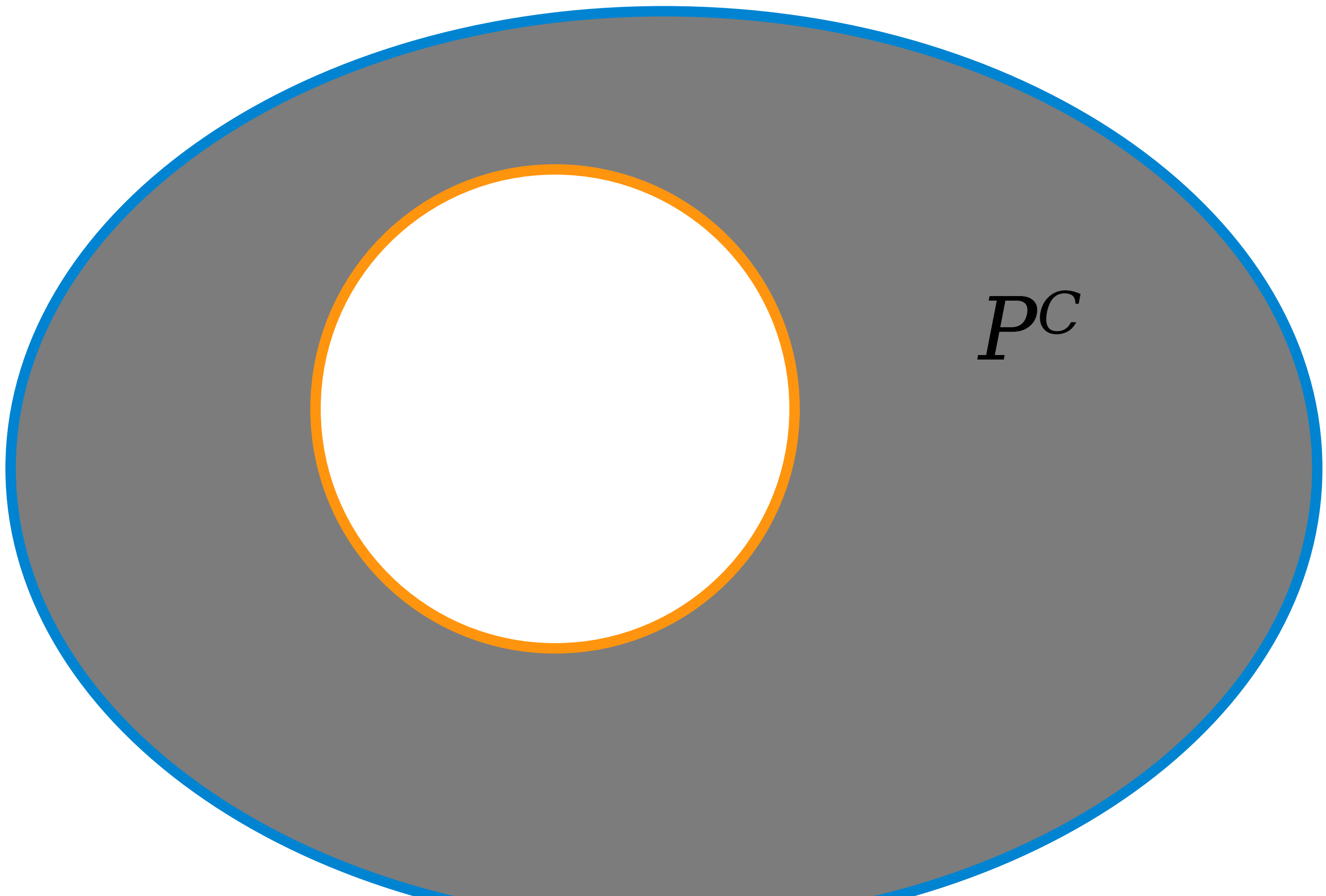
Breakouts: Negation Practice

A Different Perspective on Implication

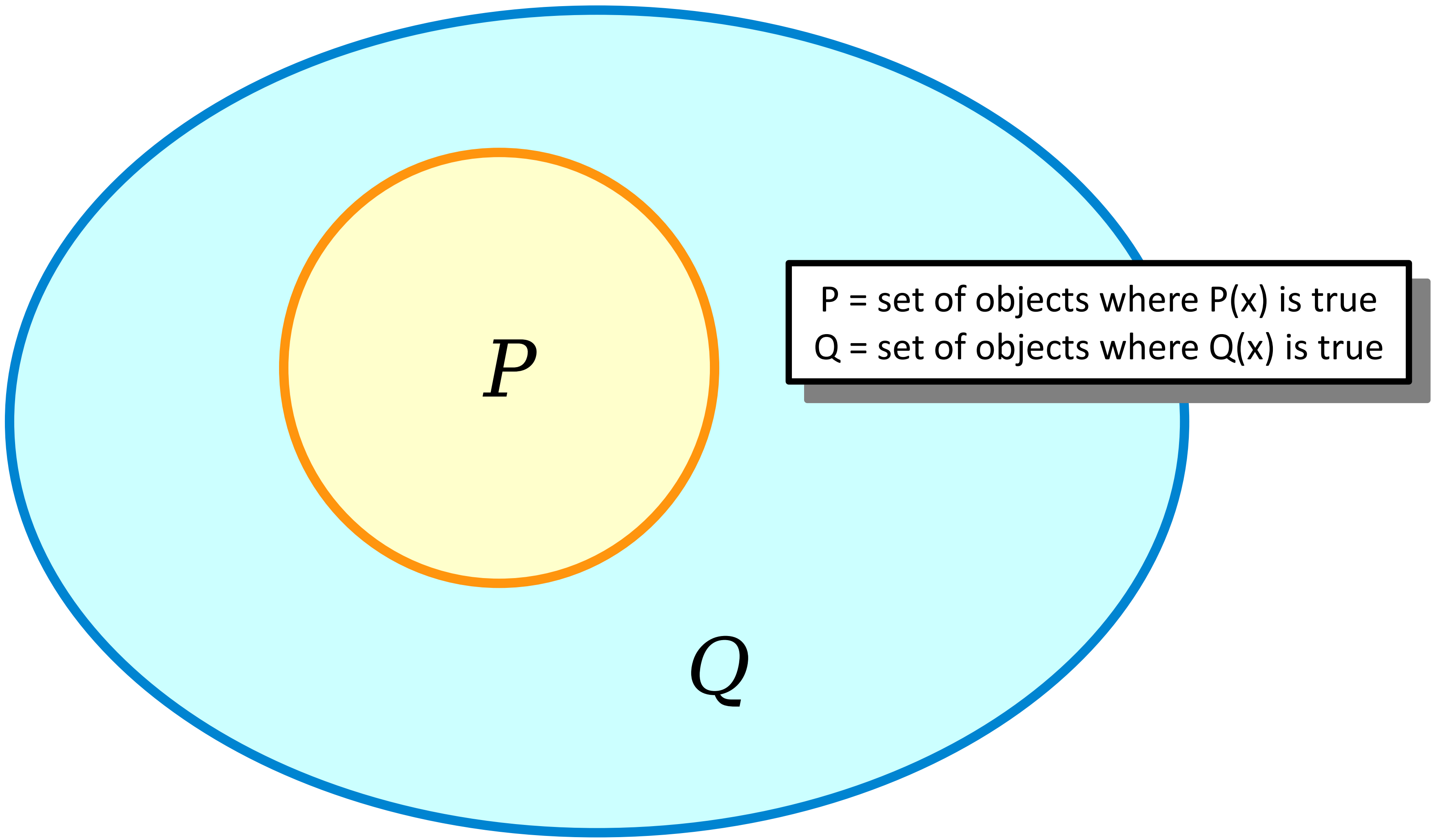
Set Complement



Set Complement

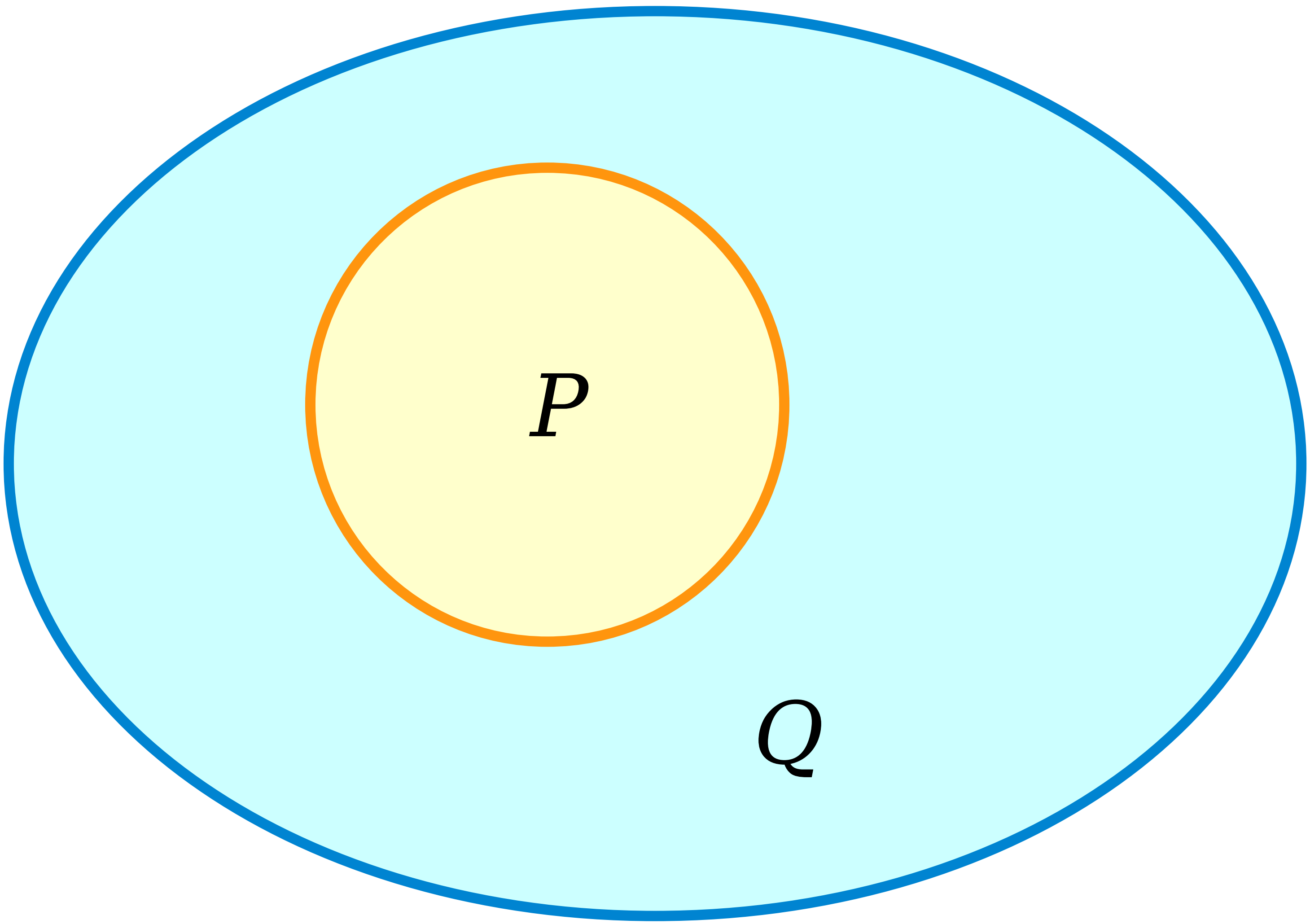


Implication, Diagrammatically



$$P \subseteq Q$$

Implication, Diagrammatically



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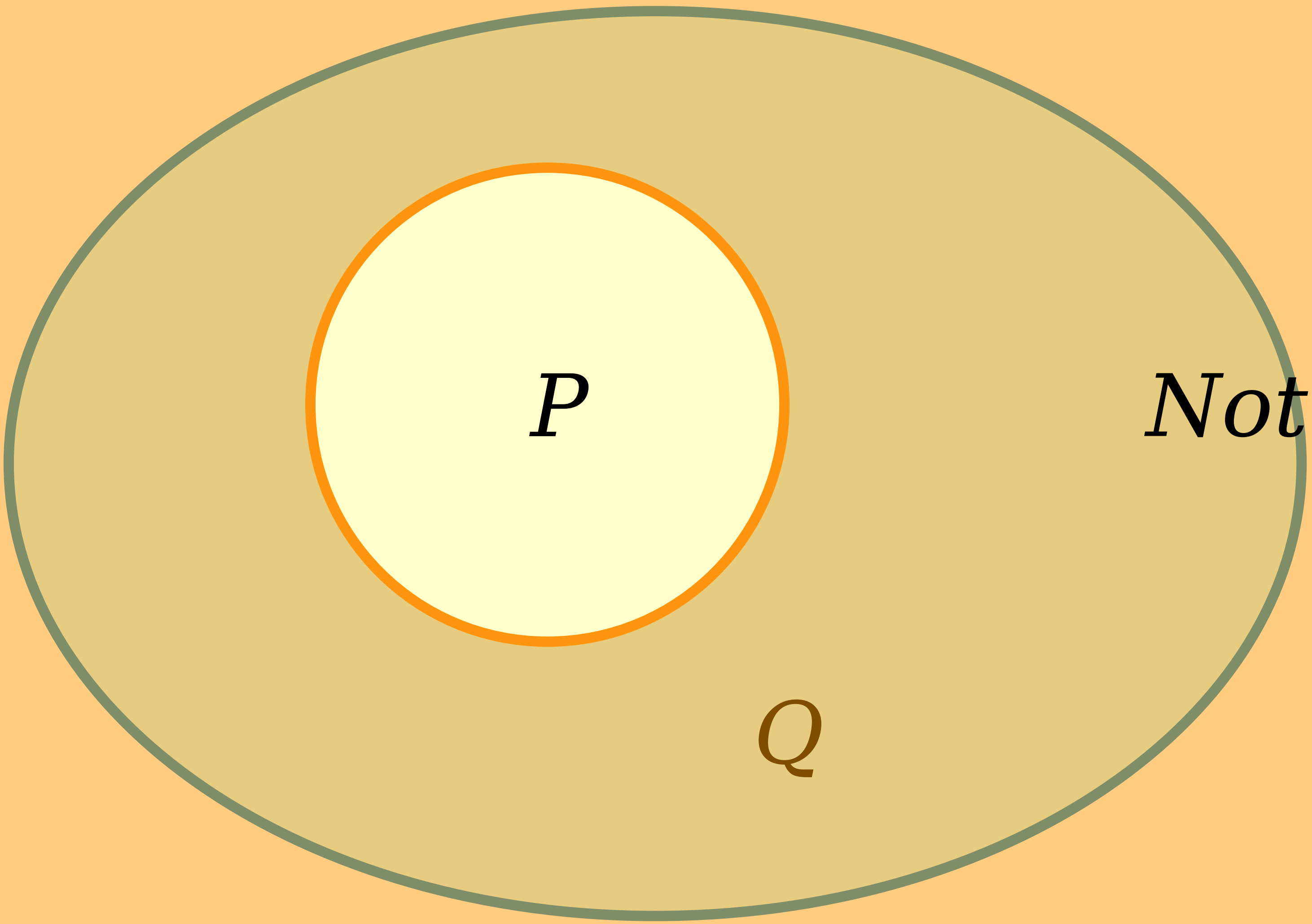

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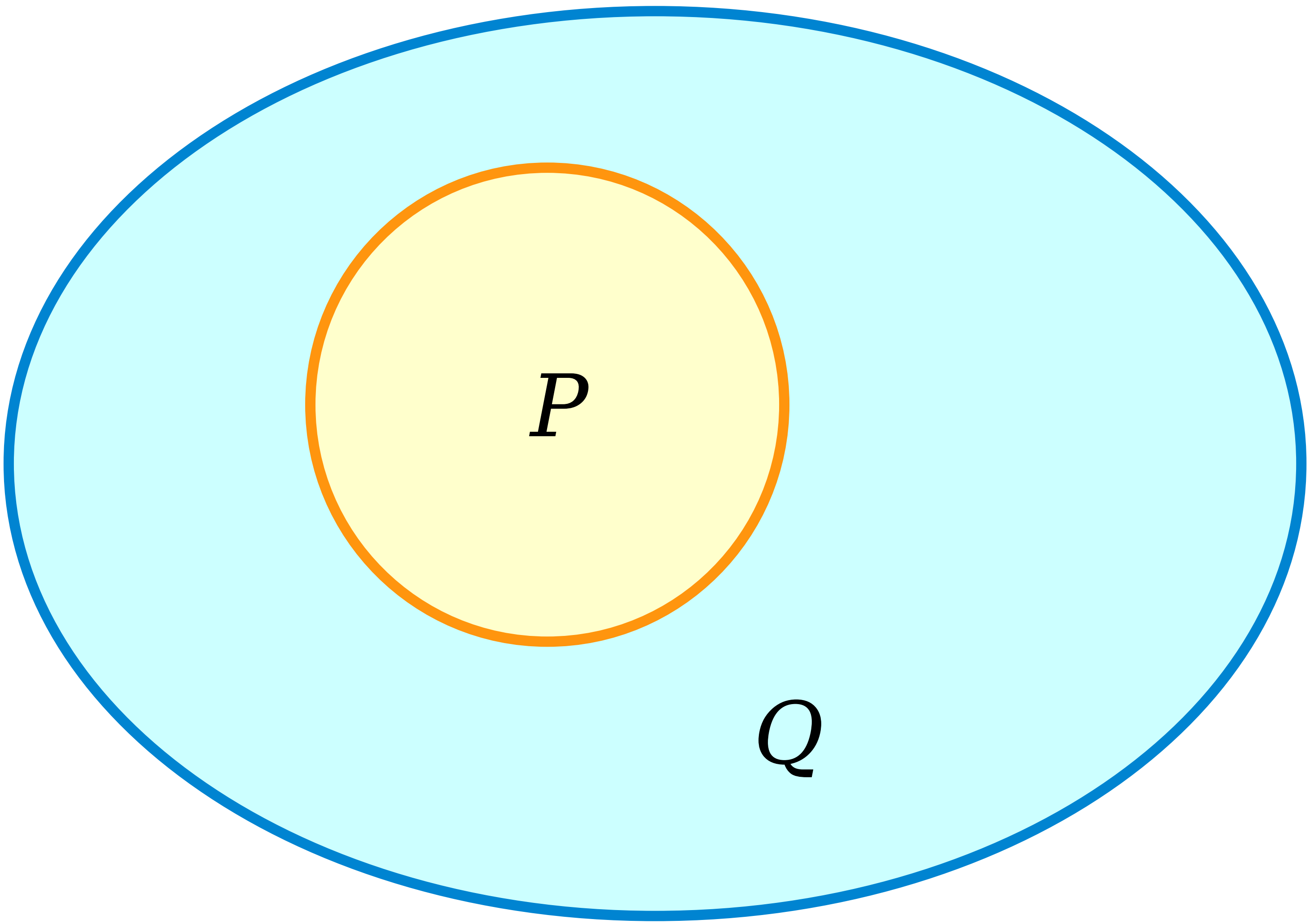

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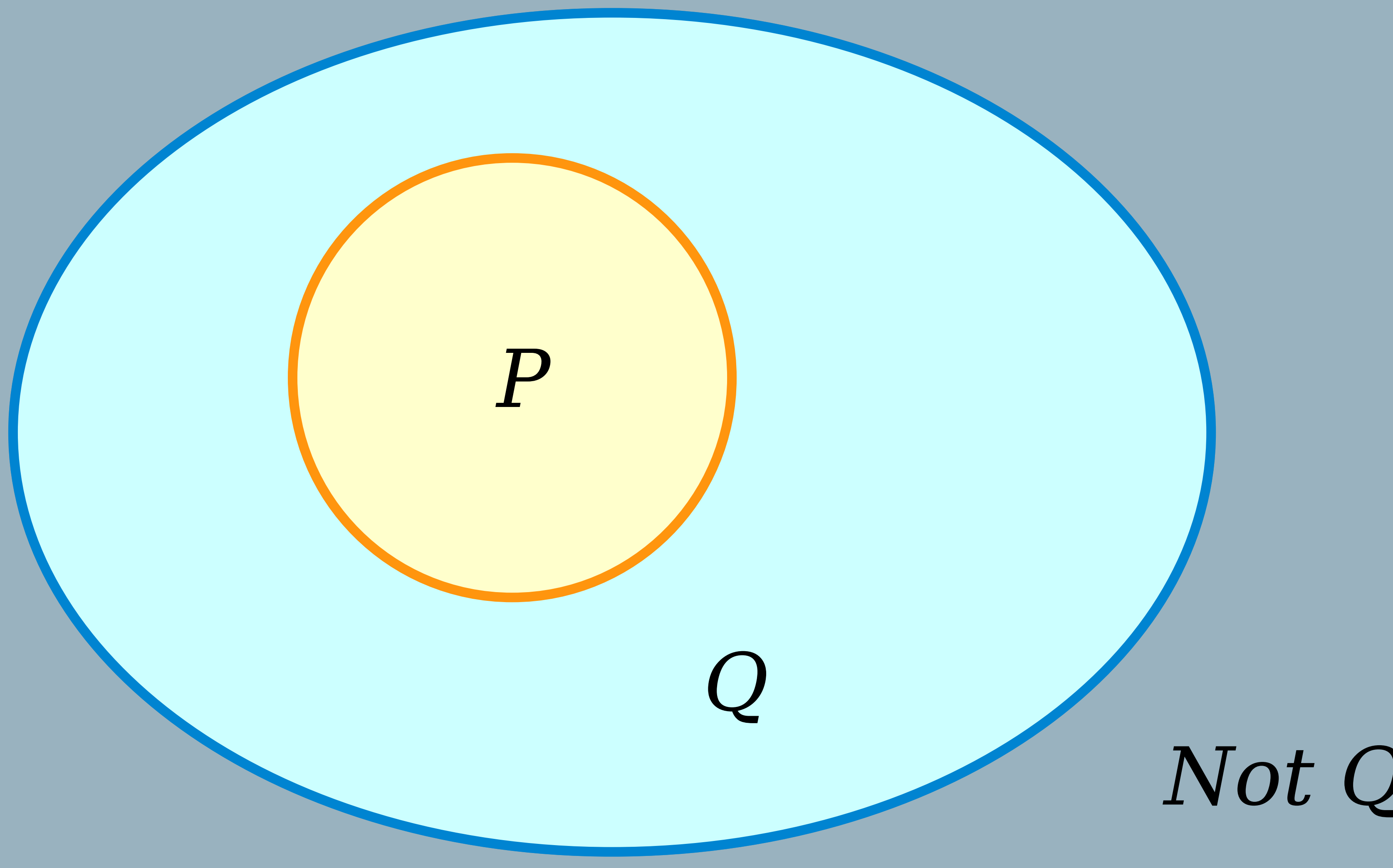

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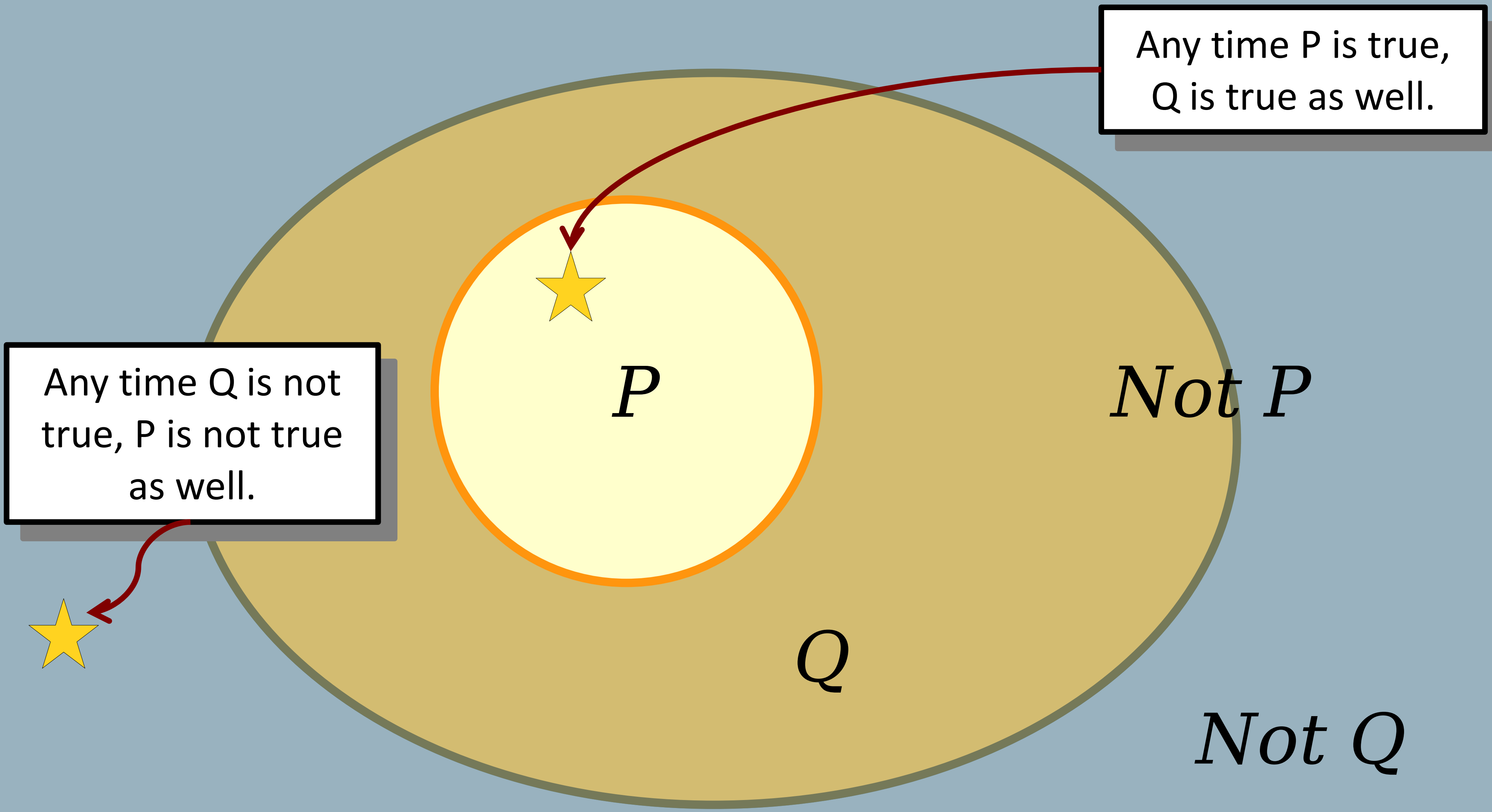


$Q \subseteq P$

Proof by Contrapositive

The Contrapositive

The ***contrapositive*** of the implication "If P, then Q" is the implication "If Q is false, then P is false."

For example:

- "If it's a puppply, then I love it."
- Contrapositive: "If I don't love it, then it's not a puppy."

Another example:

- "If I store cat food inside, then angry raccoons won't steal my cat food."
- Contrapositive: "If angry raccoons stole my cat food, then I didn't store it inside."

To prove the statement

"if *P* is true, then *Q* is true,"

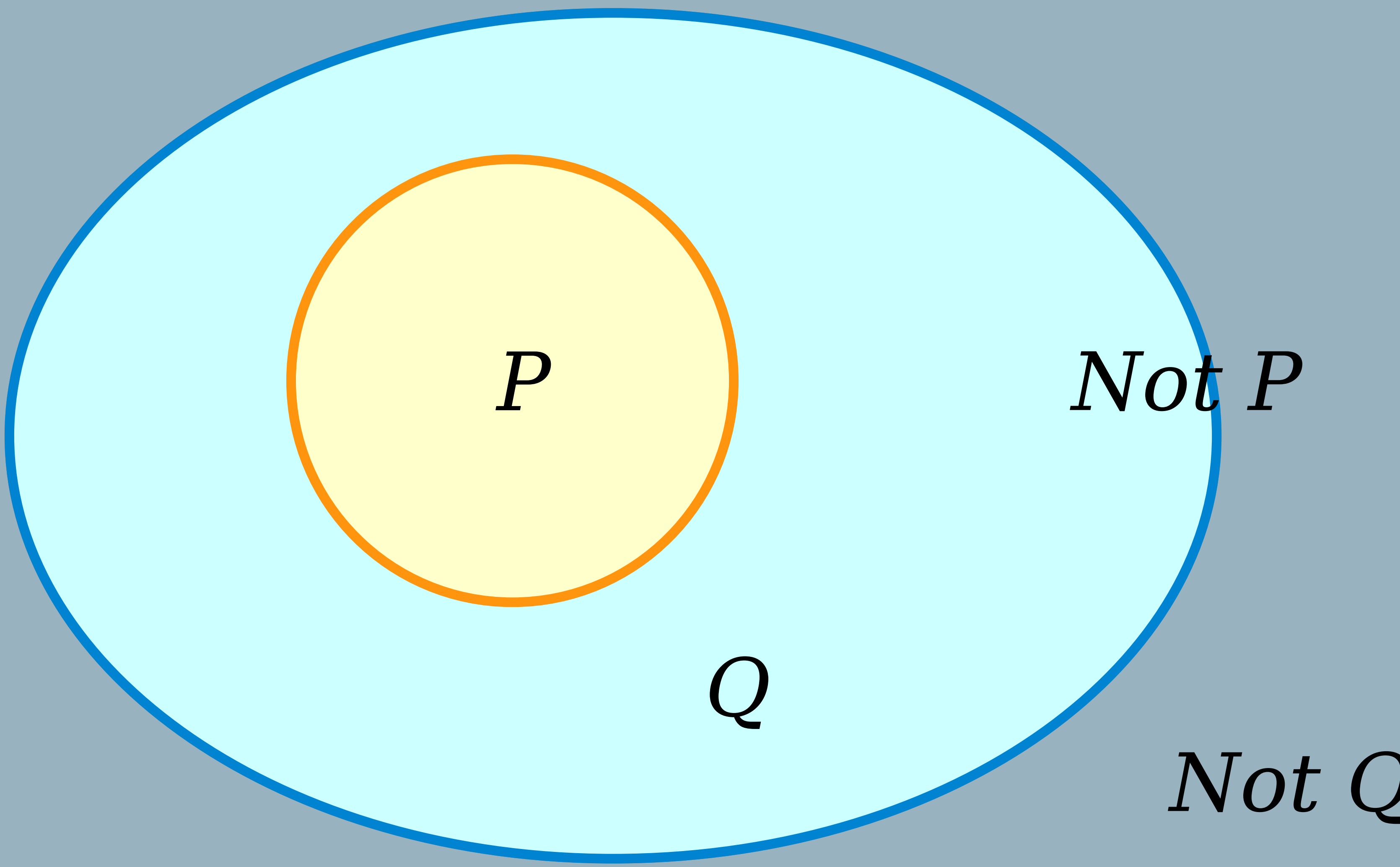
you can choose to instead prove the equivalent statement

"if *Q* is false, then *P* is false,"

if that seems easier.

This is called a ***proof by contrapositive***.

Implication, Diagrammatically



Theorem: For any $n \in \mathbb{Z}$, if n^2 is even, then n is even.

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This is a courtesy to the reader and says "heads up! we're not going to do a regular old-fashioned direct proof here."

Theorem: For any $n \in \mathbb{Z}$, if n^2 is even, then n is even.

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What is the contrapositive of the statement

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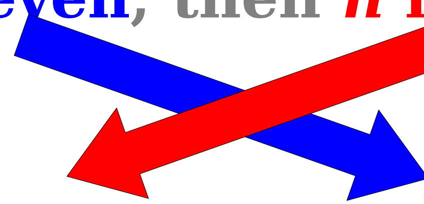

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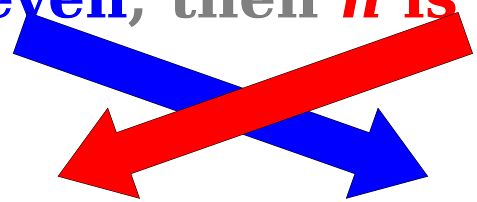

If n is odd, then n^2 is odd.

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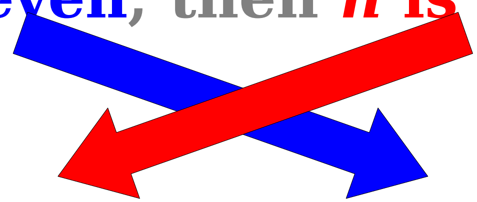

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Here, we're explicitly writing out the contrapositive. This tells the reader what we're going to prove. It also acts as a sanity check by forcing us to write out what we think the contrapositive is.

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We've said that we're going to prove this new implication, so let's go do it! The rest of this proof will look a lot like a standard direct proof.

Theorem: For any $n \in \mathbb{Z}$, if n^2 is even, then n is even.

Proof: We will prove this by contrapositive by showing that if n is odd, then n^2 is odd.

Let n be an arbitrary odd integer.

Theorem: For any $n \in \mathbb{Z}$, if n^2 is even, then n is even.

Proof: We will prove this by contrapositive by showing that if n is odd, then n^2 is odd.

Let n be an arbitrary odd integer. Since n is odd, there is some integer k such that $n = 2k + 1$.

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Let n be an arbitrary odd integer. Since n is odd, there is some integer k such that $n = 2k + 1$. Squaring both sides of this equality and simplifying gives the following:

$$n^2 = (2k + 1)^2$$

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$$\begin{aligned} n^2 &= (2k + 1)^2 \\ &= 4k^2 + 4k + 1 \end{aligned}$$

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From this, we see that there is an integer m (namely, $2k^2 + 2k$) such that $n^2 = 2m + 1$.

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$$n^2 = 4k^2 + 4k + 1$$

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Proof: We will prove this by contrapositive by showing that if n is odd, then n^2 is odd.

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The general pattern here is the following:

1. Start by announcing that we're going to use a proof by contrapositive so that the reader knows what to expect.

2. Explicitly state the contrapositive of what we want to prove.

3. Go prove the contrapositive.

$$n^2 = 2(2k^2 + 2k) + 1.$$

From this, we see that there is an integer m (namely, $2k^2 + 2k$) such that $n^2 = 2m + 1$. Therefore, n^2 is odd. ■

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Biconditionals

The previous theorem, combined with what we saw on Wednesday, tells us the following:

For any integer n, if n is even, then n^2 is even.

For any integer n, if n^2 is even, then n is even.

These are two different implications, each going the other way.

We use the phrase ***if and only if*** to indicate that two statements imply one another.

For example, we might combine the two above statements to say

.for any integer n: n is even if and only if n^2 is even.

Proving Biconditionals

To prove a theorem of the form

***P* if and only if *Q*,**

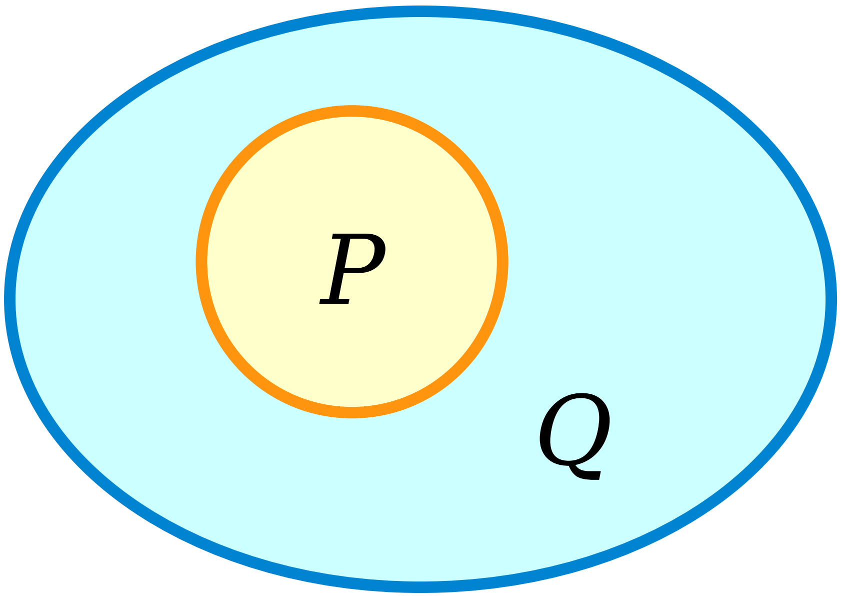
you need to prove two separate statements.

- First, that if P is true, then Q is true.
- Second, that if Q is true, then P is true.

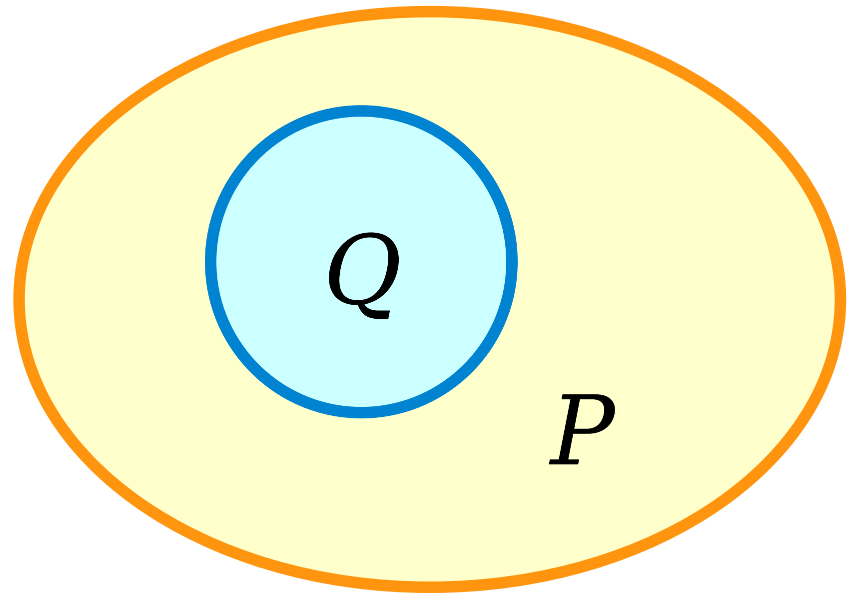
You can use any proof techniques you'd like to show each of these statements.

In our case, we used a direct proof for one and a proof by contrapositive for the other.

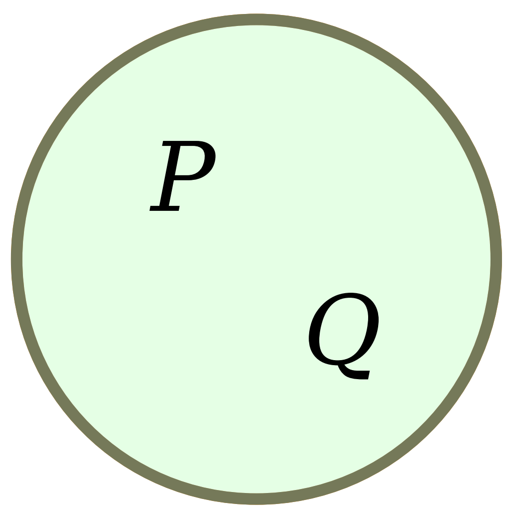
Biconditionals, Diagrammatically



If P, then Q



If Q, then P



P if and only if Q

Time-Out for Announcements!

Handouts

There are *five* total handouts released today:

- Mathematical Vocabulary
- Guide to Indirect Proofs
- Ten Techniques to Get Unstuck
- Proofwriting Checklist
- ***Problem Set One***

Be sure to read over these; there's a lot of really important information in there!

Announcements

Problem Set 1 goes out today!

- ***Checkpoint*** due Sunday, June 28th at 11:59PM.
- Grade determined by attempt rather than accuracy. It's okay to make mistakes – we want you to give it your best effort, even if you're not completely sure what you have is correct.
- We will get feedback back to you with comments on your proof technique and style.
- The more effort you put in, the more you'll get out.

Remaining problems due Thursday, July 3 at 11:59PM.

Feel free to email us with questions, stop by office hours, or ask questions on Campuswire!

Submitting Assignments

- All assignments should be submitted through GradeScope.
- The programming portion of the assignment gets submitted separately from the written component.
- The written component ***must*** be typed up in LaTeX; handwritten solutions don’t scan well and get mangled in GradeScope. LaTeX is a useful tool to learn.
- Summary of the late policy:
  - Everyone has *three* 48-hour late periods.
  - Late periods can't be used on checkpoints.
  - Nothing may be submitted more than 48 hours past the due date.

  Because submission times are recorded automatically, we're strict about the submission deadlines.
- ***Very good idea:*** Leave at least two hours buffer time for your first assignment submission, just in case something goes wrong.
- ***Very bad idea:*** Wait until the last minute to submit.

Working in Pairs

- You can work on the problem sets individually or in pairs.
- Each person/pair should only submit a single problem set. In other words, if you're working in a pair, you and your partner should agree who will make the submission.
- Full details about the problem sets, collaboration policy, and Honor Code can be found in Handout 04 and Handout 05.

A Note on the Honor Code

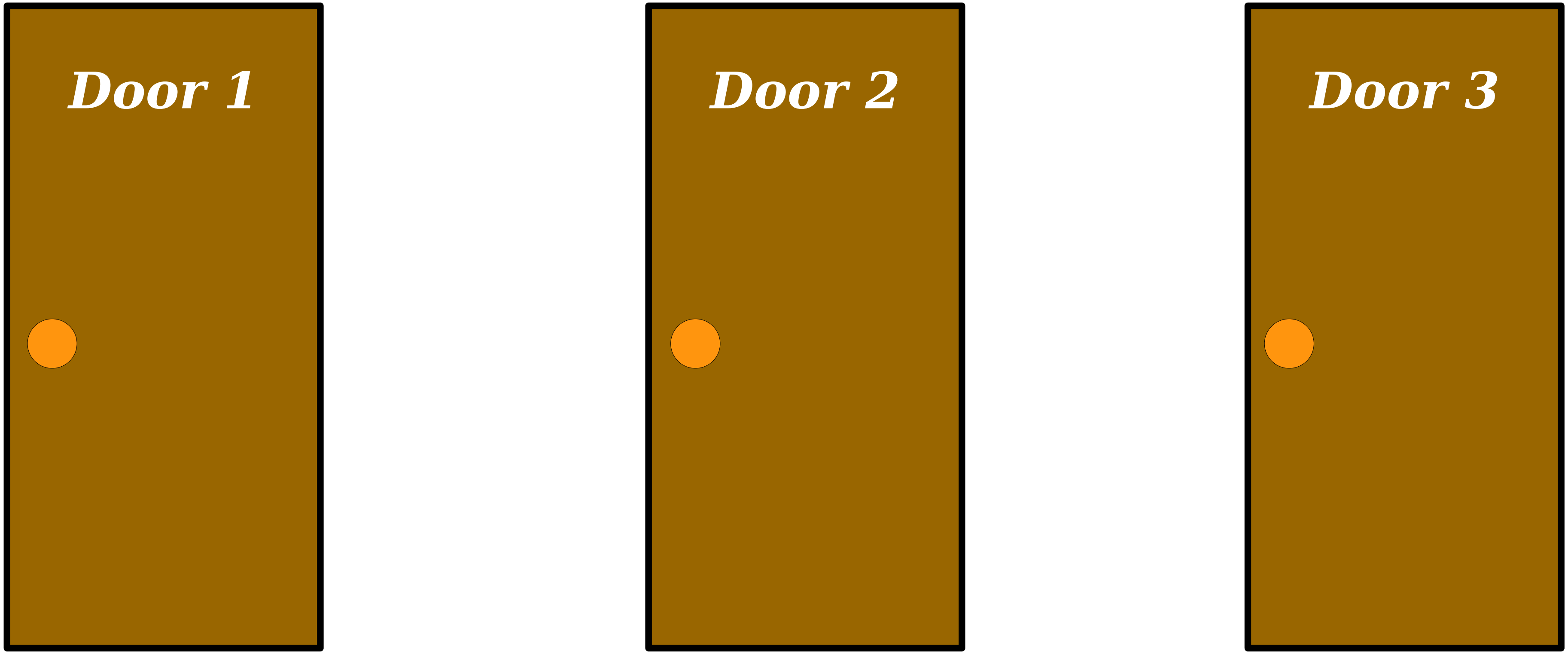
Office hours have started!

Schedule is available
on the course website.

Back to CS103!

Proof by Contradiction

There's something hidden behind one of these doors. Which door is it hidden behind?



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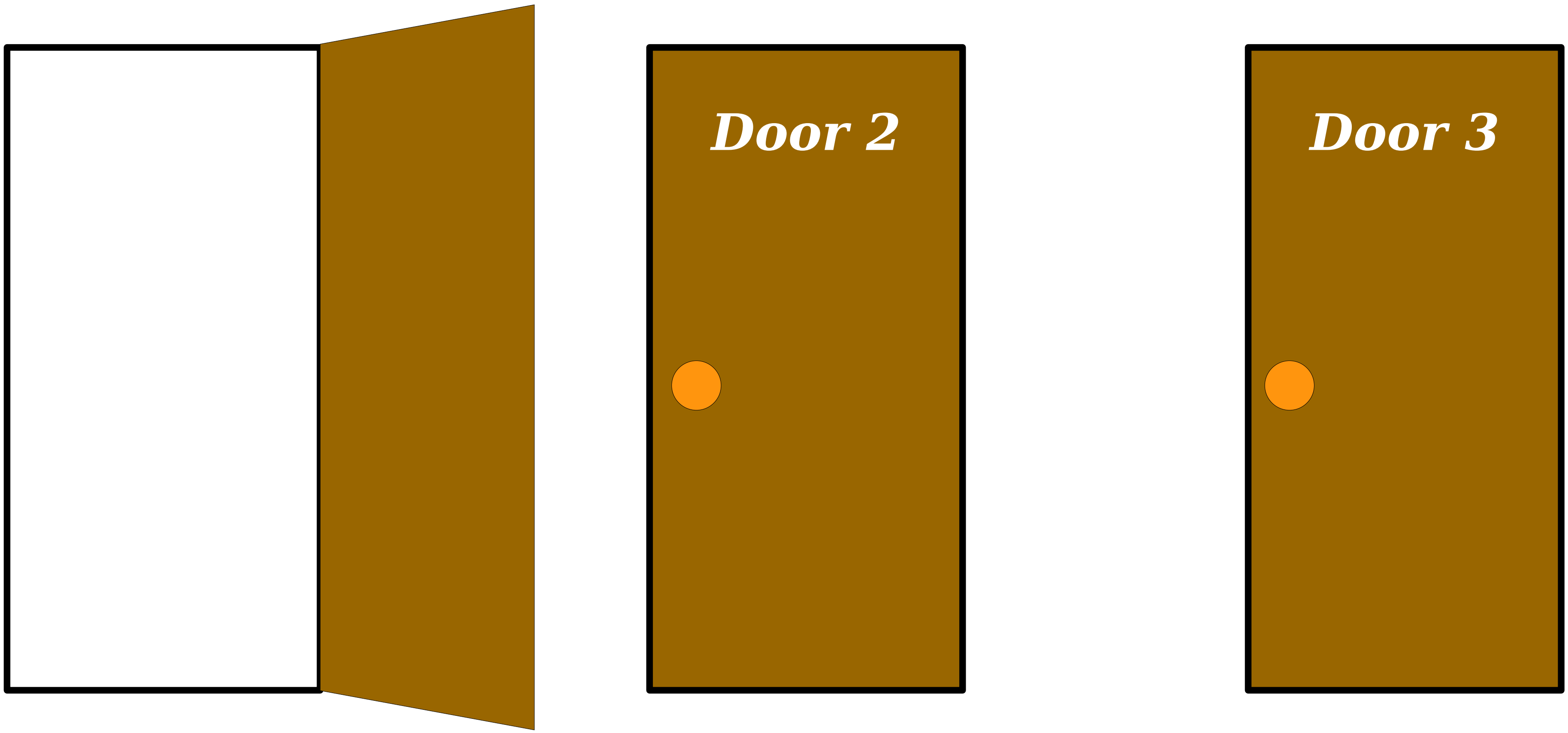

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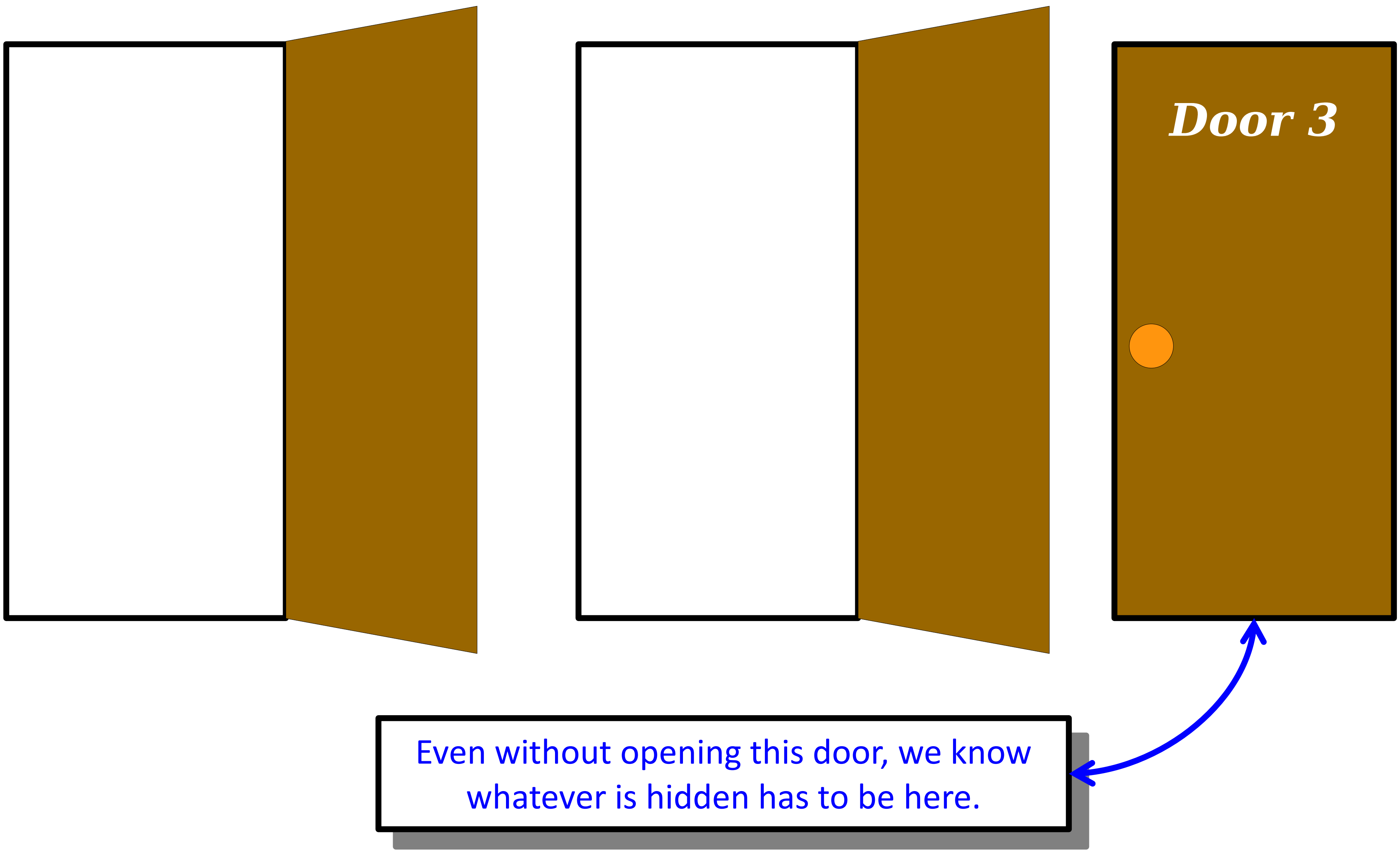

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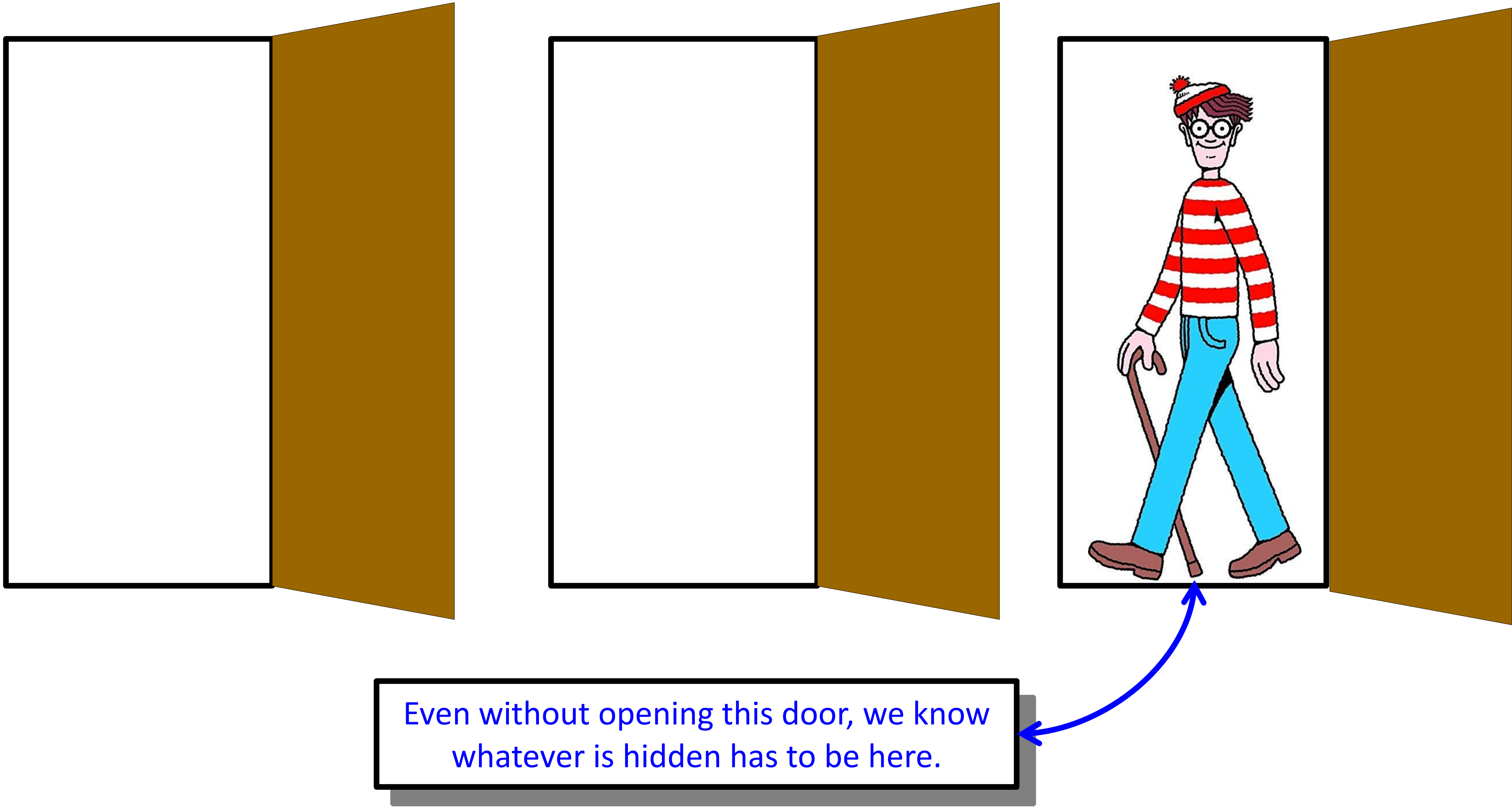
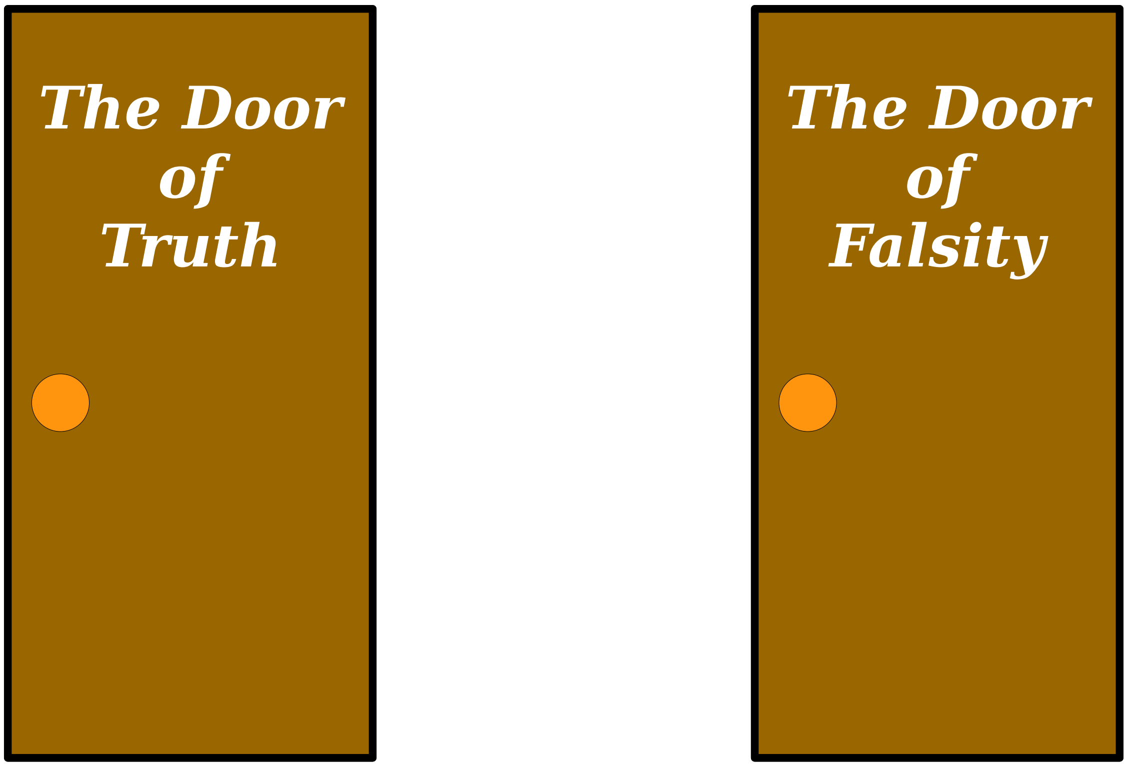


Every statement in mathematics is either true or false.
If statement P is not false, what does that tell you?



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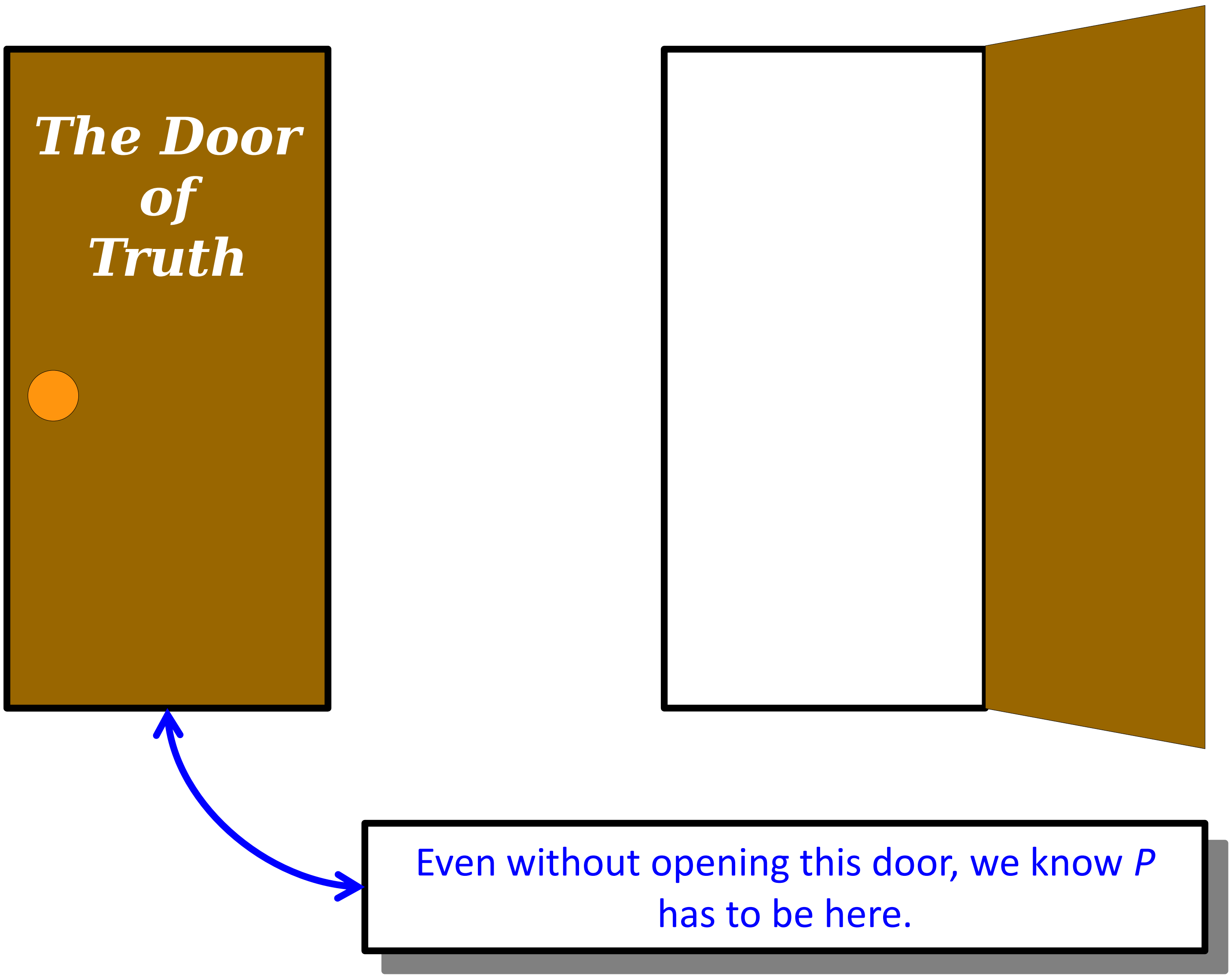
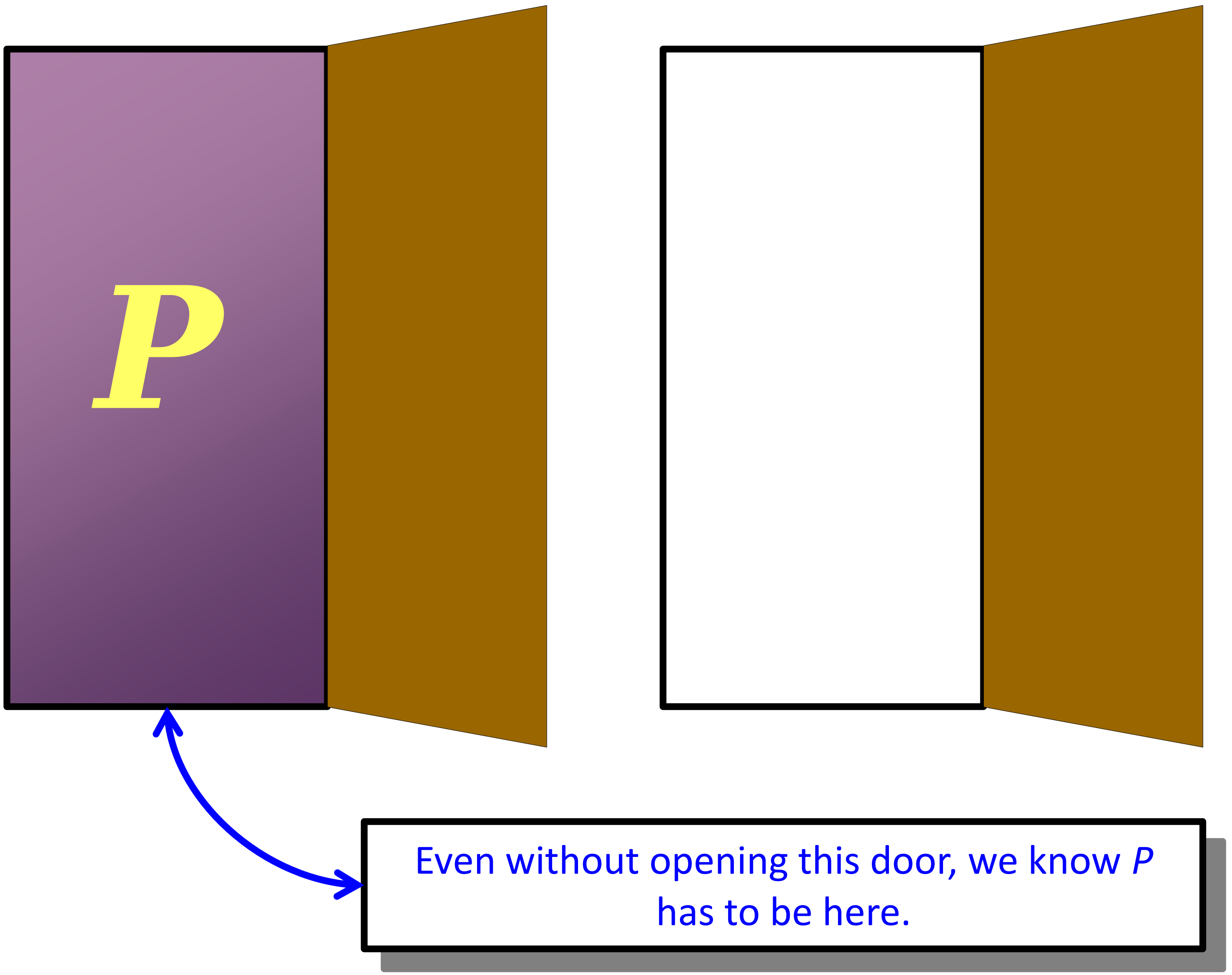


Every statement in mathematics is either true or false.
If statement P is* not false*, what does that tell you?



A ***proof by contradiction*** shows that some statement P is true by showing that it cannot be false.

"When you have eliminated the impossible, whatever remains, however improbable, must be the truth."

- Sherlock Holmes

Proof by Contradiction

To prove a statement P is true using a proof by contradiction, do the following:

- Make the assumption that P is ***false***.
- Beginning with this assumption, use logical reasoning to conclude something that is clearly impossible.
- For example, that $1 = 0$, that $x \in S$ and $x \notin S$, etc.
- Conclude that P cannot be false, so P must be true.

An Example: ***Set Cardinalities***

Set Cardinalities

We've seen sets of many different cardinalities:

- $|\varnothing| = 0$
- $|\{1, 2, 3\}| = 3$
- $|\{ n \in \mathbb{N} \mid n < 137\}| = 137$
- $|\mathbb{N}| = \aleph_0$.

These span from the finite up through the infinite.

Question: Is there a "largest" set? That is, is there a set that's bigger than every other set?

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One option: **"there *is* a largest set."**

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Proof: Assume for the sake of contradiction that there is a largest set; call it S.

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Notice that we're announcing

1. that this is a proof by contradiction, and
2. what, specifically, we're assuming.

This helps the reader understand where we're going. Remember – proofs are meant to be read by other people!

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The three key pieces:

1. Say that the proof is by contradiction.
2. Say what you are assuming is the negation of the statement to prove.
3. Say you have reached a contradiction and what the contradiction means.

In CS103, please include all these steps in your proofs!

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Proving Implications

To prove the implication

"If P is true, then Q is true."

you can use these three techniques:

Direct Proof.

- Assume P is true, then prove Q is true.

Proof by Contrapositive.

- Assume Q is false, then prove that P is false.

Proof by Contradiction.

- … what does this look like?

Theorem: For any integer n, if n^2 is even, then n is even.

What is the negation of our theorem?

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$$n = 2k + 1. \tag{1}$$

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Recap: Negating Implications

To prove the statement

“For any *x*, if *P*(*x*) is true, then *Q*(*x*) is true”

by contradiction, we do the following:

- Assume this entire purple statement is false.
- Derive a contradiction.
- Conclude that the statement is true.

What is the negation of the above purple statement?

“There is an *x* where
***P*(*x*) is true and *Q*(*x*) is false”**

If you want to prove this by contradiction...	***...assume this.***
All P's are Q's.	Some P is not a Q.
No P's are Q's.	Some P is a Q.
Some P's are Q's.	All P's are not Q's.
Some P is not a Q.	All P's are Q's.
If P is true, then Q is true.	P is true, but Q is not true.
P is true and Q is true.	P is false, or Q is false, or both are false.
P is true or Q is true, or both are true.	P is false and Q is false.

Breakout Practice:
Proofs by Contradiction and Contrapositive.

What We Learned

What's an implication?

- It's statement of the form "if *P*, then *Q*," and states that if *P* is true, then *Q* is true.

How do you negate formulas?

- It depends on the formula. There are nice rules for how to negate universal and existential statements and implications.

What is a proof by contrapositive?

- It's a proof of an implication that instead proves its contrapositive.
- (The contrapositive of "if *P*, then *Q*" is "if not *Q*, then not *P*.")

What's a proof by contradiction?

- It's a proof of a statement *P* that works by showing that *P* cannot be false.

Next Time

Mathematical Logic

- How do we formalize the reasoning from our proofs?

Propositional Logic

- Reasoning about simple statements.

Propositional Equivalences

- Simplifying complex statements.